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Unbiased Estimator as A Solution to Integral Equations

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§1. Introduction

This is an expository article of [3]. First of all we shall begin with introducing some notations. Let $D_0 := \mathbb{R}^3 \setminus \{0\}$, and we put $\mathbb{R}_+ := [0, \infty)$ for simplicity. For every pair $\alpha, \beta \in \mathbb{C}^3$, the symbol $\alpha \cdot \beta$ denotes the inner product of them, and we define $e_x := x/|x|$ for every $x \in D_0$. Let us now consider the following deterministic nonlinear integral equation:

$$
\begin{aligned}
e^{\lambda t|x|^2} u(t,x) = u_0(x) &+ \frac{\lambda}{2}\int_0^t \mathrm{d}s\ e^{\lambda s|x|^2} \int p(s,x,y;u)n(x,y)\mathrm{d}y \\
&+ \frac{\lambda}{2}\int_0^t e^{\lambda s|x|^2} f(s,x)\mathrm{d}s, \qquad \text{for} \quad \forall (t,x) \in \mathbb{R}_+ \times D_0, \qquad (1)
\end{aligned}
$$

where $u \equiv u(t,x)$ is an unknown function : $\mathbb{R}_+ \times D_0 \to \mathbb{C}^3$, $\lambda > 0$, and u_0 : $D_0 \to \mathbb{C}^3$ is the initial data such that $u(t,x)|_{t=0} = u_0(x)$. Moreover, $f(t,x)$: $\mathbb{R}_+ \times D_0 \to \mathbb{C}^3$ is a given function satisfying $f(t,x)/|x|^2 =: \tilde{f} \in L^1(\mathbb{R}_+)$ for each $x \in D_0$. The integrand p in (1) is given by $p(t,x,y;u) = u(t,y) \cdot e_x\{u(t,x-y) - e_x(u(t,x-y) \cdot e_x)\}$. We assume that the integral kernel $n(x,y)$ is bounded and measurable with respect to $\mathrm{d}x \times \mathrm{d}y$. On the other hand, we consider a Markov kernel $K : D_0 \to D_0 \times D_0$. Actually, for every $z \in D_0$, $K_z(\mathrm{d}x, \mathrm{d}y)$ lies in the space $\mathcal{P}(D_0 \times D_0)$ of all probability measures on a product space $D_0 \times D_0$. When the kernel k is given by $k(x,y) = i|x|^{-2}n(x,y)$, then we define K_z as a Markov kernel satisfying that for any positive measurable function $h = h(x,y)$ on $D_0 \times D_0$,

$$
\iint h(x,y)K_z(\mathrm{d}x, \mathrm{d}y) = \int h(x, z-x)k(x,z)\mathrm{d}x. \qquad (2)
$$

In [2] we have reported our new result, which asserts the existence and uniqueness of probabilistic solutions to the nonlinear integral equation (1). That is to say, we have succeeded in deriving a probabilistic representation of the solutions to (1) by employing the star-product functional. As a matter of fact, the solution $u(t,x)$ can be expressed as the expectation of a star-product functional, which is nothing but a probabilistic solution constructed by making use of suitable branching particle systems and branching models. The following is nothing but a probabilistic representation of the solution in terms of tree-based star-product functional with weight (u_0, f):

$$M_{\bigstar}^{\langle u_0,f\rangle}(\omega) = \prod \bigstar_{[x_{\tilde{m}}]}\Xi_{m_2.m_3}^{m_1}[u_0, f](\omega). \qquad (3)$$

For details of the definition, see the Section 4 in [2]. On the other hand, $M_*^{\langle U,F\rangle}(\omega)$ denotes the associated $*$-product functional with weight (U, F), which is indexed by the nodes (x_m) of a binary tree. Here $U = U(x)$ (resp. $F = F(t, x)$) is a non-negative measurable function on D_0 (resp. $\mathbb{R}_+ \times D_0$) respectively, and also that $F(\cdot, x) \in L^1(\mathbb{R}_+)$ for each x. Indeed, in construction of the $*$-product functional, the product in question is taken as ordinary multiplication $*$ instead of the star-product $\bigstar$ in the definition of star-product functional. Then we have:

THEOREM 1. (see also [3]) *Suppose that* $|u_0(x)| \leqslant U(x)$ *for* $\forall x$ *and* $|\tilde{f}(t, x)| \leqslant F(t, x)$ *for* $\forall t, x$, *and also that for some* $T > 0$ ($T >> 1$ *sufficiently large*), $E_{T,x}[M_*^{\langle U,F\rangle}(\omega)] < \infty$, a.e.-$x$ *Then there exists a* (u_0, f)*-weighted tree-based star* $\bigstar$*-product functional* $M_{\bigstar}^{\langle u_0,f\rangle}$ (ω), *indexed by a set of node labels accordingly to the tree structure which a binary critical branching process* $Z^{K_x}(t)$ *determines. Furthermore, the function* $u(t, x) = E_{t,x}[M_{\bigstar}^{\langle u_0,f\rangle}(\omega)]$ *gives a unique solution to the integral equation* (1). *Here* $E_{t,x}$ *denotes the expectation with respect to a probability measure* $P_{t,x}$ *as the time-reversed law of* $Z^{K_x}(t)$.

§2. Unbiased estimator and linear integral equation

In this section we shall think of the following linear integral equation for a while:

$$u(x) = \int_{\bar{D}} u(y)k(x, \mathrm{d}y) + g(x), \qquad x \in D; \qquad u(y)|_{\partial D} = \varphi(y), \quad y \in \partial D, \quad (4)$$

where D is an open domain in a certain metric space (M, ρ). Here we suppose that a function φ is bounded on the boundary ∂D. Let $\xi_0 = x$, $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be a Markov chain with a phase space $\bar{D}$ and a transition function $p(y, \mathrm{d}z)$ being convergent to one on the boundary ∂D: namely, $\xi_n \to \xi_\infty \in \partial D$ holds a.s. and $p(y, \{z\}) = 1$ holds for any $z \in \partial D$. We assume further that (i) there exists the Radon-Nikodym derivatives $k_p(y, z) = k(y, \mathrm{d}z)/p(y, \mathrm{d}z)$; (ii) when we put $\zeta = \sup_k \prod_{i=0}^{k} |k_p(\xi_i, \xi_{i+1})|$, then for any $x \in D$, $\mathbb{P}_x(\zeta < +\infty) = 1$ and $\mathbb{E}_x[\zeta^2] < +\infty$; (iii) an infinite product $\prod_{i=0}^{\infty} k_p(\xi_i, \xi_{i+1})$ converges $\mathbb{P}_x$-a.s for any $x \in D$; (iv) $\sum_{k=0}^{\infty} \mathbb{E}_x[g^2(\xi_k)] < \infty$.

When we define a random variable η as $\eta = \prod_{i=0}^{\infty} k_p(\xi_i, \xi_{i+1})\varphi(\xi_\infty) + \sum_{k=0}^{\infty} \prod_{i=0}^{k-1} k_p(\xi_i, \xi_{i+1})g(\xi_k)$, then η is an unbiased estimator for one of the solutions to (4). According to the theory by Ermakov, it is proven that if g is a uniformly continuous function on D, then the function $u(x) = \mathbb{E}_x[\eta]$ is a unique bounded solution to the problem (4). Furthermore, as a matter of fact, it is also proved that

$$\eta^*_{N_0(\varepsilon)} = \prod_{i=0}^{N_0(\varepsilon)-1} k_p(\xi_i, \xi_{i+1})\varphi(\xi^*_{N_0(\varepsilon)}) + \sum_{k=0}^{N_0(\varepsilon)} g(\xi_k) \sum_{i=0}^{k-1} k_p(\xi_i, \xi_{i+1}) \tag{5}$$

is a biased estimator of $u(x)$ with $N_0(\varepsilon) := \min\{n : \xi_n \in \Gamma_\varepsilon\}$ where Γ_ε is an inner ε-neighborhood of the boundary ∂D for a sufficiently small positive number $\varepsilon > 0$, and the random element $\xi^*_{N_0(\varepsilon)}$ lies in the boundary ∂D and $\rho(\xi_{N_0(\varepsilon)}, \xi^*_{N_0(\varepsilon)}) \leqslant \varepsilon$. In fact, the estimator $\eta^*_{N_0(\varepsilon)}$ is realizable as far as $\mathbb{E}_x[N_0(\varepsilon)] < \infty$ holds.

§3. Principal result

Our result (Theorem 1) stated in Section 1 provides with a invaluable example that a probabilistic solution to (1) has been constructed, but its analytical solution having an explicit representation has not known yet. Then the next question might have been asked by someone else: "since you have an explicit solution, you can simulate behaviors of the solution, is it correct?" In this moment we are not confident that our solution (3) (also the expectation of (3)) is a realizable unbiased estimator. A next thing to do should be to investigate whether a realizable unbiased estimator for (1) exists or not. Then the result on construction of an unbiased estimator of the solutions to (1) will be introduced. In fact, an answer to the aforementioned question is as follows.

THEOREM 2. (cf. [3]) *Let $\{\xi_k\}_{k=0}^{\infty}$ be a Markov chain with a phase space $(\mathbb{D}, \mathcal{G})$ and with a transition function $p(x, dy)$. Then there exist a suitable sequence $\{\tau_m\}$ of random variables and a proper functional $M^*(\xi) = M^*(\tau_m, \{\xi_k\})$ of $\{\tau_m\}$* and *$\{\xi_k\}$ such that a random quantity $\zeta = M^*(\xi)$ is a realizable unbiased estimator of the solutions to the integral equation* (1), i.e., *in other words, the function*

$$u(t, x) := \hat{E}_{t,x}[M^*(\xi)] = \hat{E}_{t,x}[M^*(\tau, \{\xi_k\})] \tag{6}$$

satisfies (1), *where $\hat{E}_{t,x}$ is the expectation with respect to $Q_{t,x}$, and $Q_{t,x}$ is a probability measure on $(\mathbb{D}, \mathcal{G})$ induced by the probability law $P_{t,x}$ in Theorem 1.*

§4. Sketch of the proof of theorem 2

The purpose of this section is to prove Theorem 2, our main result stated in Section 3. That is, we shall construct a realizable unbiased estimator $\zeta = M^*(\xi)$ of the solutions to our deterministic nonlinear integral equation (1). Most known Monte Carlo algorithms are based on the simulation of realizable processes, and it is also clear that the Markov property of these processes is very important. It is certain that construction of *unbiased estimators* is a significant issue, while it is necessary for the process $\zeta = g(\eta)$, a kind of representation, to be *realizable.*

4.1. Markov chain and characterization equation

We shall start with defining of a branching Markov chain $\{\xi_n\}_n$. First of all, we put

$$\mathbb{D} := \bigoplus_{n=0}^{\infty} D_0^n, \qquad D_0^0 = \Delta \notin D_0^n \quad (n = 1, 2, \dots), \tag{7}$$

and let D_0 be equipped with the σ-field $\mathcal{G}$ generated by a natural σ-algebra $\mathcal{B}(D_0)$ and a single-point set $\{\Delta\}$, where Δ is called an absorbing state in the theory of Markov processes. Inspired by Ermakov's theory, for each $x \in D_0$ we define $\{p_n\}$ by a probability measure $p(x, \mathrm{d}y) \in \mathcal{P}(\mathbb{D}, \mathcal{G})$ such that $p_n(x, A) := p(x, A)$ for $A \subset D_0^n$ and $g(x) = p(x, \{\Delta\})$ as a measurable function. Here $p(x, \mathrm{d}y)$ is called a transition function of Markov chain $\{\xi_n\}$. We assume that for every $A \in \mathcal{G}$, $p(x, A)$ is a $\mathcal{B}(D_0)$-measurable function. Moreover, we set $r_n(x) := p_n(x, D_0^n)$, and $q_n(x, \mathrm{d}y) := p_n(x, \mathrm{d}y)/r_n(x)$, $n \geq 1$, so that, $q_n(x, D_0^n) = 1$. A family of measures $\{p(x, \mathrm{d}y), x \in D_0\}$ will be called a branching law. Practically, we set $G_A(x) = p_2(x, A) = \frac{1}{2}$ for $A \in D_0^2$ and $p_0(x, D_0^0) = \frac{1}{2}$, and let $\{\tau_m\}$ be a sequence of random variables such that under $\xi_{p(m)} = x_m$, they are mutually independent and have an exponential distribution with parameter $\lambda |x_m|^2$, when $p(m) : \mathcal{V} \to \mathbb{N}_0$ is a lexicographically numbering mapping. At each time τ_m, random particles $\xi_{p(m)}$ are governed by

$$p_2^{[x_{m''}]}(\xi_{p(m)}, \xi_{p(m')}, A) = K_{x_{m''}}((\mathrm{d}x_m, \mathrm{d}x_{m'}) \in \Phi^{-1}(A) \times \Phi^{-1}(A)), \quad \forall A \in D_0^2 \tag{8}$$

with $|m| = |m'| = \ell$ when $|m''| = \ell - 1$ for $m, m', m'' \in \mathcal{V}$, where Φ is a mapping from D_0 to $\mathbb{D}$. Then a process η_t may be described in the following way: a process stays for a random time τ_1 at the initial state $\eta_{p(1)} = \xi_{p(1)} = x_1$, then if jumps into states $\xi_{p(11)}$ and $\xi_{p(12)}$ where the process stays for a time $\tau_{p(11)}$ and $\tau_{p(12)}$ respectively, and so forth. Here note that $\tau_1(\omega) = \inf\{t \geq 0; \xi_{p(1)} \neq \xi_{p(11)}$ and $\xi_{p(1)} \neq \xi_{p(12)}\}$ and also that for $a > 0$, $x_0 \in D_0$ and a measurable set $A \subset \mathbb{D}$

$$\mathbb{E}_x[e^{-a\tau_1},\ \eta_{\tau_1} \in A | \eta_{\tau_1 -}] = p(\eta_{\tau_1 -}, A)\mathbb{E}_x[e^{-a\tau_1} | \xi_{\tau_1 -}] \tag{9}$$

holds on the set $\{\tau_1 < +\infty\}$. Set $F(x) = \prod_{i=1}^n f(x_i)$ for $x = (x_1, x_2, \dots, x_n)$ and $F(\Delta) = 0$.

LEMMA 3. *Set $\tau := \max\{k : \xi_k \neq \Delta\}$. Then the function $v(x) = \mathbb{E}_x[F(\xi_\tau), \tau < \infty]$ satisfies the equation*

$$v(x) = g(x)f(x) + \iint_{D_0^2} p_2(x, \mathrm{d}y) v(y_1) v(y_2) \quad \textit{with} \quad v_0 = gf \textit{ and } y = (y_1, y_2). \tag{10}$$

Proof. Since $\tau \geq 1$, then by a property of conditional expectation we can get

$$v(x) = \mathbb{E}_x[F(\xi_\tau), \tau < \infty] = \mathbb{E}_x[F(\xi_\tau), \tau = 1] + \mathbb{E}_x[F(\xi_\tau), 1 < \tau < \infty]$$
$$= g(x)f(x) + \sum_{n=1}^{\infty} \int_{D_0^n} p_n(x, \mathrm{d}y^n) \prod_{i=1}^{n} v(y_i) \tag{11}$$

with $y^n = (y_1, y_2, \ldots, y_n)$. Because we made use of the Markov property in the above calculation. The definition of $p_n(x, \mathrm{d}y)$ reminds us of the validity that the last integral term in the above (11) can be rewritten into the double integral $\iint_{D_0^2} p_2(x, \mathrm{d}y)v(y_1)v(y_2)$ with $y = (y_1, y_2)$. Thus we attain the expression (10). □

When the corresponding quantity consists of an infinite number of Markov chains, the estimator generated by them itself is not realizable since it depends upon the infinite trajectories of the Markov chain. Generally speaking, the branching process is a purely discontinuous one, that is, the changing of a state of the process occurs only at a branching instant. Hence, the branching process must be realizable, in particular the process considered in this section itself must be realizable at least.

4.2. Construction of unbiased estimator

In this section, based upon the Markov chain described in the previous section, we are going to construct an unbiased estimator of the solutions to integral equation (1), that is realizable. We shall write the number of the elements in $\mathcal{N}(\omega)$ of a realized tree structure Ω by $N_e(\omega)$, and shall rewrite it and make an ordinary numbering in a lexicographical manner for those quantities $\Xi^{m_1}_{m_2,m_3}(\omega)$, $\Theta^{m_i}(\omega)$ (see [2]), tagged with labels $m \in \mathcal{V}$, so that, we define

$$\varphi_{p(m)}(\xi) \equiv \varphi_{p(m)}(t_{p(m)}, \xi_{p(m)}, \omega) = \begin{cases} \tilde{f}(t_m(\omega), x_m(\omega)) & \text{if} \quad \omega \in N_+(\omega), \\ u_0(x_m(\omega)) & \text{if} \quad \omega \in N_-(\omega), \end{cases} \tag{12}$$

where we put $t_{p(m)} := \tau_m$. For simplicity, for $\varphi_{p(m)}(\xi) = (\varphi^1_{p(m)}(\xi), \varphi^2_{p(m)}(\xi), \varphi^3_{p(m)}(\xi))$, we write it as $\varphi^i_k(\xi)$; $i = 1, 2, 3$; $k = 1, 2, \ldots, N_e(\omega)$ by the abuse of notation. Based upon our theory stated and explained in [2], we write the objective functional of Markov chain as

$$M^*(\xi) \equiv M^*(\tau_m, \{\xi_n\}) = \prod_{i=1}^{N_e(\omega)} \bigstar_{[\xi_i]} \, \Xi_{(i)}(\varphi_{p(m)}(\xi)). \tag{13}$$

This symbolic expression indicates the $\bigstar$-product of $N_e(\omega)+1$ pieces of functionals $\Xi_{(i)}(\varphi_{p(m)}(\xi))$ of $(\tau_m, \{\xi_n\})$, with pivoting point ξ_i, that is realized explicitly by a finite combination of ordinary functions $\tilde{f}$ and u_0. Each $\bigstar_{[\xi_i]}$-operation between terms $\Xi_{(i)}(\varphi_{p(m)}(\xi))$'s should be succeedingly executed in a lexicographical manner with respect to the original label $m \in \mathcal{V}$, just described as in [2] (see also [3]). For brevity's sake we illustrate the typical case by a simple example, to see how it goes or what it looks like. Let us now consider a simple case of $\Xi_{(i)}(\varphi_{p(m)}(\xi))$ involved with $\varphi_{p(m)}(\xi)$ and $\varphi_{p(m')}(\xi)$ with pivoting $\xi_{p(m'')}$. Then we have immediately

$$\begin{aligned}\Xi_{(i)}(\varphi_{p(m)}(\xi)) &= i\alpha(\xi_{p(m'')}) \times \beta(\varphi_{p(m)}(\xi), \xi_{p(m'')}) \times \gamma(\xi_{p(m'')}, \varphi_{p(m')}(\xi)) \\ &= \frac{i\sum_{j=1}^{3} \varphi_k^j(\xi)\xi_{k''}^j}{\sqrt{(\xi_{k''}^1)^2 + (\xi_{k''}^2)^2 + (\xi_{k''}^3)^2}} \times \left\{ \frac{\sum_{j=1}^{3} \xi_{k''}^j \varphi_{k'}^j(\xi)}{(\xi_{k''}^1)^2 + (\xi_{k''}^2)^2 + (\xi_{k''}^3)^2} \cdot \xi_{k''} - \varphi_{k'}(\xi) \right\} \end{aligned} \tag{14}$$

where we put $p(m) = k$, $p(m') = k'$ and $p(m'') = k''$, and both particles with labels m and m' belong to the same ℓ-th generation of descendants since we have $|m| = |m'| = \ell$ when $|m''| = \ell - 1$.

We define a new probability measure $Q_{t,x}$ on $(\mathbb{D}, \mathcal{G})$ by $Q_{t,x}(A) := P_{t,x}(\Phi^{-1}(A))$, for every $A \in \mathbb{D}$, for each $(t, x) \in [0, T] \times D_0$, when a mapping $\Phi : D_0 \to \mathbb{D}$ is given. Then we denote by $\hat{E}_{t,x}$ the expectation with respect to the probability measure $Q_{t,x}$. By employing the conditional expectation, we obtain, for every $0 \leqslant t \leqslant T$ and $x \in E_c$, with the event F_0 indicating no branching

$$\begin{aligned} u(t,x) &= \hat{E}_{t,x}[M^*(\xi)] = \hat{E}_{t,x}[M^*(\tau_m, \{\xi_n\})] \\ &= \hat{E}_{t,x}[M^*(\tau_m, \{\xi_n\}), \tau_1 \leqslant 0] + \hat{E}_{t,x}[M^*(\tau_m, \{\xi_n\}), \tau_1 > 0,\ F_0] \\ &\qquad + \hat{E}_{t,x}[M^*(\tau_m, \{\xi_n\}), \tau_1 > 0,\ \text{branching occurs}]. \end{aligned} \tag{15}$$

We need the following series of lemmas.

LEMMA 4. *For every* $0 \leqslant t \leqslant T$ *and* $x \in E_c$, *we have*

$$\hat{E}_{t,x}[M^*(\tau_m, \{\xi_n\}, \tau_1 \leqslant 0] = u_0(x) \cdot \exp\{-\lambda |x|^2 t\}. \tag{16}$$

Proof. In this case, branching time never lives in $[0, T]$ with $T > 0$. Hence, clearly a simple computation yields at once to

$$\begin{aligned} &\hat{E}_{t,x}[M^*(\tau_m, \{\xi_n\}), \tau_1 \leqslant 0] = \hat{E}_{t,x}[\prod_{i=1}^{N_e(\omega)} \bigstar_{[\xi_i]} \Xi_{(i)}(\varphi_{p(m)}(\xi)), \tau_1 \leqslant 0] \\ &= \hat{E}_{t,x}[\varphi_{p(\phi)}(t_{p(\phi)}, \xi_{p(\phi)}, \omega), \tau_1 \leqslant 0] = u_0(x) \cdot P_{t,x}(t_\phi \leqslant 0) = u_0(x) \cdot e^{-\lambda |x|^2 t}, \end{aligned} \tag{17}$$

where we have used a similar argument as in Section 7 of [3]. □

LEMMA 5. *For every* $0 \leqslant t \leqslant T$ *and* $x \in E_c$, *we have*

$$\hat{E}_{t,x}[M^*(\tau_m, \{\xi_n\}), \tau_1 > 0,\ \text{no branching}] = \int_0^t \frac{\lambda |x|^2 \tilde{f}(s,x)}{2} \cdot \exp\{-\lambda |x|^2 (t-s)\} ds.$$

Proof. As to this case, $\tau = t - s$ is distributed in the exponential distribution with parameter $\lambda|x|^2 > 0$ when $t_\phi = s \in [0, T]$. Taking it into account the situation that extinction occurs at some time between 0 and t, we can obtain easily

$$\hat{E}_{t,x}[M^*(\tau_m, \{\xi_n\}), \tau_1 > 0, \text{ no branching}] = \hat{E}_{t,x}[\varphi_{p(\phi)}(t_{p(\phi)}, \xi_{p(\phi)}, \omega), \tau_1 > 0, \ F_0]$$
$$= \hat{E}_{t,x}[\tilde{f}(\tau_\phi, \xi_{p(\phi)}, \omega), \tau_1 > 0, \ F_0] = \frac{1}{2} \times \int_0^t \mathrm{d}s \lambda |x|^2 e^{-\lambda|x|^2(t-s)} \times \tilde{f}(s, x),$$

since by the condition the branching occurs at time $t_\phi = s$, under the probability $\frac{1}{2}$ (which comes from the constraint $p_0(x, D_0^0) = \frac{1}{2}$). This finishes the proof. □

Likewise, we can get the following lemma which play an essential role in construction of unbiased estimator of the solutions to integral equation (1).

LEMMA 6. *We have*

$$\hat{E}_{t,x}[M^*(\tau_m, \{\xi_n\}), \tau_1 > 0, \text{ branching occurs}] = \int_0^t \mathrm{d}s \frac{\lambda|x|^2}{2} \exp\{-\lambda|x|^2(t-s)\} \cdot$$
$$\times \iint_{D_0^2} \hat{E}_{t,x}[M^*(\xi)] \bigstar_{[x]} \hat{E}_{t,x}[M^*(\xi)] p_2^{[x]}(\xi_{p(m)}, \xi_{p(m')}, \mathrm{d}y \times \mathrm{d}z) \tag{18}$$

with $y = x_m(\omega)$ *and* $z = x_{m'}(\omega)$, *for every* $0 \leqslant t \leqslant T$ *and* $x \in E_c$.

Proof. By the Markov property applied at time t_ϕ, we have

$$\hat{E}_{t,x}[M^*(\tau_m, \{\xi_n\}), \tau_1 > 0, \text{ branching occurs}] = \hat{E}_{t,x}[\hat{E}_{t_\phi, x(t_\phi)}[M^*(\tau_m, \{\xi_n\}), \tau_1 > 0]].$$

If branching occurs at time $t_\phi = s$ $(0 \leqslant s \leqslant t)$, then the time interval $\tau = t - s$ is exponentially distributed with parameter $\lambda|x|^2 > 0$, so that, the time variable s ranges over $(0, t)$, and its probability for branching occurrence is given by an integral with respect to s over $(0, t)$. In addition to that, the number $\frac{1}{2}$ comes out from the restraint $G_A(x) = \frac{1}{2}$, which means the probability that the parent particle at $x(t_\phi)$ produces two offsprings that jump into the locations, for instance, y and z. Then, by virtue of independence of Markov families, the expectation separates into a form of product of two expectations at branching time s just like $\hat{E}_{s,y}[M^*(\xi)] \bigstar_{[x]} \hat{E}_{s,z}[M^*(\xi)]$, and those offspring particles are governed by the probability measure $p_2^{[x]}(\xi_{p(m)}, \xi_{p(m')}, \mathrm{d}y \times \mathrm{d}z)$. Consequently, summing up all of them, we can deduce

$$\hat{E}_{t,x}[M^*(\tau_m, \{\xi_n\}), \tau_1 > 0, \text{ branching occurs}] = \frac{1}{2} \times \int_0^t \mathrm{d}s \lambda |x|^2 \cdot \exp\{-\lambda|x|^2(t-s)\} \cdot$$
$$\times \iint_{D_0^2} \hat{E}_{s,y}[M^*(\xi)] \bigstar_{[x]} \hat{E}_{s,z}[M^*(\xi)] p_2^{[x]}(\xi_{p(m)}, \xi_{p(m')}, \mathrm{d}y \times \mathrm{d}z)$$

with $y = x_m(\omega)$ and $z = x_{m'}(\omega)$, for every $0 \leqslant t \leqslant T$ and $x \in E_c$. □

Combining the above-obtained results: Lemmas 4, 5 and 6, we obtain therefore

$$
\begin{aligned}
u(t,x) &= \hat{E}_{t,x}[M^*(\tau_m,\{\xi_n\})] \\
&= u_0(x)\cdot\exp\{-\lambda|x|^2 t\} + \int_0^t \frac{\lambda|x|^2\tilde{f}(s,x)}{2}\exp\{-\lambda|x|^2(t-s)\}\mathrm{d}s \\
&+ \int_0^t \mathrm{d}s\frac{\lambda|x|^2}{2}e^{-\lambda|x|^2(t-s)}\iint_{D_0^2}\hat{E}_{s,y}[M^*(\xi)]\bigstar_{[x]}\hat{E}_{s,z}[M^*(\xi)]p_2^{[x]}(\xi_{p(m)},\xi_{p(m')},\mathrm{d}y\times\mathrm{d}z).
\end{aligned}
\tag{19}
$$

Recall that $u(s,y)=\hat{E}_{s,y}[M^*(\xi)]$ and $u(s,z)=\hat{E}_{s,z}[M^*(\xi)]$. And besides, from (8) the transition function in (19) can be rewritten into the Markov kernel $K_x(\mathrm{d}y,\mathrm{d}z)$. Recall (10). Then the last term in (19) is changed into a form

$$
\int_0^t \mathrm{d}s\frac{\lambda|x|^2}{2}e^{-\lambda|x|^2(t-s)}\int p(s,x,y;u)n(x,y)\mathrm{d}y,
$$

where we made use of the integral formula (2). This implies that $u(t,x) = \hat{E}_{t,x}[M^*(\tau_m,\{\xi_n\})]$ satisfies the nonlinear integral equation (1), that is to say, $\zeta = M^*(\xi)$ gives a realizable unbiased estimator of the solutions to (1). This completes the proof of Theorem 2.

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